

IRRATIONAL STABLE COMMUTATOR LENGTH IN FINITELY PRESENTED GROUPS

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ABSTRACT. We give examples of finitely presented groups containing elements with irrational (in fact, transcendental) stable commutator length, thus answering in the negative a question of M. Gromov. Our examples come from 1-dimensional dynamics, and are related to the generalized Thompson groups studied by M. Stein, I. Lioussé and others.

1. INTRODUCTION

Let G be a group, and let a be an element of the commutator subgroup, which we always denote by $[G, G]$. The *commutator length* of a , which we denote $cl(a)$, is defined to be the minimum number of commutators whose product is equal to a . That is,

$$cl(a) = \min\{n \mid a = [b_1, c_1] \cdots [b_n, c_n]\}$$

$cl(a)$ is a subadditive function, so the limit of $\frac{cl(a^n)}{n}$, as $n \rightarrow \infty$, exists.

Definition 1.1. $G, a \in [G, G]$, the *stable commutator length* of a , denoted by $scl(a)$, is defined to be

$$scl(a) = \lim_{n \rightarrow \infty} \frac{cl(a^n)}{n}$$

Set $scl(a) = \infty$, if no power of a is in $[G, G]$.

Commutator length and stable commutator length in groups have long been studied, often under the name of the *genus* problem. If $G = \pi_1(V)$ for some aspherical space V , and γ is a loop in V representing the conjugacy class of an element a , then the commutator length of a is the minimal genus of a surface S for which there is a map $f : S \rightarrow V$ taking ∂S to γ . More generally, given a class $H \in H_2(V, \gamma)$, one can ask for the least genus immersed surface in the relative class H . Stabilizing, one obtains a norm on $H_2(V, \gamma)$. If G is finitely presented, $H_2(V, \gamma)$ is finitely dimensional, and one can try to minimize this stable norm on the subspace which is the preimage of the class of $[\gamma]$ under the boundary map $H_2(V, \gamma) \rightarrow H_1(\gamma)$. This infimum is the stable commutator length of a .

M. Gromov (in [10] 6.C₂) asked the question of whether such a stable norm on $H_2(V, \gamma)$, or in our content, the stable commutator length in a finitely presented group, is always rational, or more generally, algebraic. The purpose of this note is to give simple (and even natural) examples which show that the stable commutator length in finitely presented groups can be transcendental.

We now state the contents of this note. In §2 we state the fundamental *Duality Theorem* of C. Bavard [2], which gives a precise relationship between stable commutator length and homogeneous quasimorphisms on groups. In §3 we give our examples and demonstrate that they have the desired properties, which are based on the work of M. Stein [14], D. Calegari [4] and I. Lioussé [13]. The examples are central extensions of certain (finitely generated) groups of piecewise linear homeomorphisms of the circle. C. Bavard's duality

theorem connects dynamics (rotation numbers, as studied by I. Lioussé [13]) with stable commutator length. We include an appendix presenting basic properties of rotation numbers. They are used in §2 and §3.

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2. STABLE COMMUTATOR LENGTH AND QUASIMORPHISMS

Definition 2.1. Let G be a group. A *quasimorphism* on G is a function

$$\phi : G \rightarrow \mathbb{R}$$

for which there is a constant $D(\phi) \geq 0$ such that for any $a, b \in G$, we have an inequality

$$|\phi(a) + \phi(b) - \phi(ab)| \leq D(\phi)$$

In other words, a quasimorphism is like a homomorphism up to a bounded error. The least constant $D(\phi)$ with this property is called the *defect* of ϕ .

Definition 2.2. A quasimorphism is *homogeneous* if it satisfies the additional property

$$\phi(a^n) = n\phi(a)$$

for all $a \in G$ and $n \in \mathbb{Z}$.

A homogeneous quasimorphism is a class function by its definition. Denote the vector space of all homogeneous quasimorphisms on G by $Q(G)$.

Example 2.3. Let $\widetilde{\text{Homeo}^+(S^1)} = \{f \in \text{Homeo}^+(\mathbb{R}) \mid f(x+1) = f(x) + 1\}$. It's the set consisting of all possible lifts of elements in $\text{Homeo}^+(S^1)$ under the covering projection $\pi : \mathbb{R} \rightarrow S^1$. We have the central extension:

$$0 \rightarrow \mathbb{Z} \rightarrow \widetilde{\text{Homeo}^+(S^1)} \rightarrow \text{Homeo}^+(S^1) \rightarrow 1$$

where \mathbb{Z} is generated by the unit translation and $\widetilde{p} : \widetilde{\text{Homeo}^+(S^1)} \rightarrow \text{Homeo}^+(S^1)$ is the natural projection from its definition. For $g \in \widetilde{\text{Homeo}^+(S^1)}$, define

$$\text{rot}(g) = \lim_{n \rightarrow \infty} \frac{g^n(0)}{n}$$

With this definition, rot is a homogeneous quasimorphism with defect 1. (See appendix for a proof.)

We have the following fundamental theorems of C. Bavard, which state the duality between scl and homogeneous quasimorphisms. See C. Bavard [2] or D. Calegari [5] for a reference.

Theorem 2.4 (C. Bavard).

(1) Let G be a group. Then for any $a \in [G, G]$, we have an equality

$$scl(a) = \frac{1}{2} \sup_{\phi \in Q(G)/H^1(G; \mathbb{R})} \frac{|\phi(a)|}{D(\phi)}$$

(2) *There is an exact sequence*

$$0 \rightarrow H^1(G; \mathbb{R}) \rightarrow Q(G) \rightarrow H_b^2(G; \mathbb{R}) \rightarrow H^2(G; \mathbb{R})$$

(3) *Let G be a group. Then the canonical map from bounded cohomology to ordinary cohomology $H_b^2(G; \mathbb{R}) \rightarrow H^2(G; \mathbb{R})$ is injective if and only if the stable commutator length vanishes on $[G, G]$.*

Given a group G , if $Q(G)$ is small enough, we can use (1) of Theorem 2.4 to determine scl . For the previous example $G = \widetilde{\text{Homeo}^+(S^1)}$, we have $\dim_{\mathbb{R}} Q(G) = 1$.

Proposition 2.5 (J. Barge and É. Ghys [1]). *$rot : \widetilde{\text{Homeo}^+(S^1)} \rightarrow \mathbb{R}$ is the unique homogeneous quasimorphism which sends the unit translation to 1.*

Proof: Suppose $\tau \in Q(\widetilde{\text{Homeo}^+(S^1)})$ is another such map, then we consider

$$rot - \tau : \widetilde{\text{Homeo}^+(S^1)} \rightarrow \mathbb{R}$$

which is also a homogeneous quasimorphism, and since any homogeneous quasimorphism on abelian groups, especially $\mathbb{Z} \oplus \mathbb{Z}$, must be a homomorphism, we have

$$(rot - \tau)(f_1) = (rot - \tau)(f_2)$$

if $p(f_1) = p(f_2)$. It therefore induces a homogeneous quasimorphism on $\text{Homeo}^+(S^1)$, denote it still by $rot - \tau : \text{Homeo}^+(S^1) \rightarrow \mathbb{R}$. But $\text{Homeo}^+(S^1)$ is uniformly perfect, i.e. cl is bounded ([9]), so the induced map is bounded. Since it's homogeneous, it must be a zero map, i.e. $rot = \tau$ \square

By Theorem 2.4 and Proposition 2.5 together, we have $scl(g) = \frac{1}{2}rot(g)$, for any g in $\widetilde{\text{Homeo}^+(S^1)}$. Suppose G is a subgroup of $\widetilde{\text{Homeo}^+(S^1)}$ which is uniformly perfect. Let \tilde{G} be the preimage of G in $\text{Homeo}^+(S^1)$. Then by the same argument, $scl(a) = rot(a)/2D$ for any a in \tilde{G} , where D is the defect of rot restricted to \tilde{G} .

Our goal therefore is to find the subgroups of $\text{Homeo}^+(S^1)$ which are finitely presented and uniformly perfect, and contain elements with interesting rotation numbers. As a first example, consider Thompson's well-known group of dyadic piecewise linear homeomorphisms.

Let \tilde{G} consist of piecewise linear homeomorphisms f of \mathbb{R} with the following properties:

- (1) For each point x_i of discontinuity of the derivative of f (hereafter a "break point"), both x_i and $f(x_i)$ are dyadic rational numbers (i.e. of the form $p2^q$, $p, q \in \mathbb{Z}$);
- (2) The derivatives of the restrictions of f to (x_i, x_{i+1}) are powers of 2 (i.e. of the form 2^m , $m \in \mathbb{Z}$);
- (3) f preserves dyadic rational numbers and $f(x+1) = f(x) + 1$.

The elements of \tilde{G} induce piecewise linear homeomorphisms of $S^1 \simeq \mathbb{R}/\mathbb{Z}$. The collection of these homeomorphisms is the *Thompson group* G ([15], [6]). Thompson group is simple, FP_∞ and uniformly perfect ([6], [7] and [8]), so we have (The defect is still 1. See appendix.)

$$scl(a) = \frac{1}{2}rot(a), a \in \tilde{G}$$

About the rotation numbers of elements in the Thompson group, we have

Theorem 2.6 (É. Ghys and V. Sergiescu [8]). *$rot(a)$ is rational for any $a \in \tilde{G}$.*

So scl takes only rational values on the group \tilde{G} .

3. GENERALIZED THOMPSON GROUPS

Our definition of generalized Thompson groups is from [14] by M. Stein. Let P be a multiplicative subgroup of the positive real numbers and let A be a $\mathbb{Z}P$ -submodule of the reals with $PA = A$. Choose a number $l \in A, l > 0$. Let $F(l, A, P)$ be the group of piecewise linear homeomorphisms of $[0, l]$ with finitely many break points, all in A , having slopes only in P . Similarly define $T(l, A, P)$ to be the group of piecewise linear homeomorphisms of $[0, l]/\{0, l\}$ (the circle formed by identifying endpoints of the closed interval $[0, l]$) with finitely many break points in A and slopes in P , with the additional requirement that the homeomorphisms send $A \cap [0, l]$ to itself. In these notations, $T(1, \mathbb{Z}[\frac{1}{2}], \langle 2 \rangle)$ is the Thompson group. In our study of generalized Thompson groups, we always assume that P is generated by the set of positive integers $\{n_1, n_2, \dots, n_k\}$ and $A = \mathbb{Z}[\frac{1}{n_1}, \frac{1}{n_2}, \dots, \frac{1}{n_k}]$, here $\{n_1, n_2, \dots, n_k\}$ forms a basis for P . An important theorem in studying generalized Thompson groups is the following Bieri-Strebel criterion (See [14] appendix for a proof.).

Theorem 3.1 (R. Bieri and R. Strebel [3]). *Let a, c, a', c' be elements of A with $a < c$ and $a' < c'$. Then there exists f , a piecewise linear homeomorphism of \mathbb{R} , with slopes in P and finitely many break points, all in A , mapping $[a, c]$ onto $[a', c'] \Leftrightarrow c' - a'$ is congruent to $c - a$ modulo $IP * A$.*

Here $IP * A$ is the submodule of A generated by elements of the form $(1 - p)a$, where $a \in A$ and $p \in P$. Let $d = \gcd(n_1 - 1, \dots, n_k - 1)$ and from now on, we assume that $d = 1$. In this case, $IP * A = P(\mathbb{Z}) = P\mathbb{Z} = A$, so the Bieri-Strebel criterion from Thm 3.1 is vacuously satisfied.

Take an arbitrary $f \in T = T(l, A, P)$ with the assumptions above. Choose points $a < b, c < d \in [0, l] \cap A$, $f([a, b]) = [a_1, b_1]$ such that $[c, d] \cap ([a, b] \cup [a_1, b_1]) = \emptyset$ (This can be achieved by taking $[a, b]$ small enough.). We can find piecewise linear homeomorphisms g_1 and g_2 with slopes in P and break points in A , sending $[b, c]$ to $[b_1, c]$ and $[d, a]$ to $[d, a_1]$ respectively. Construct $g \in T$ as follows.

$$g = \begin{cases} f & \text{if } x \in [a, b] \\ g_1 & \text{if } x \in [b, c] \\ id & \text{if } x \in [c, d] \\ g_2 & \text{if } x \in [d, a] \end{cases}$$

Now $f = (fg^{-1})g$, both $f \circ g^{-1}$ and g fix some nonempty open arcs.

Write $f = g_1 g_2$ where g_i 's fix some open arcs for $i = 1, 2$. For each such g_i , there exists a rotation through θ_i , $R_{\theta_i} \in T$ such that $\text{supp}(R_{\theta_i} \circ g_i \circ R_{-\theta_i}) \subseteq (0, l)$. ($R_{\theta_i} \in T$ if and only if $\theta_i \in A$ which is dense in $[0, l]$, so we only need to choose θ_i in the open arc fixed by g_i .) Write $h_i = R_{\theta_i} \circ g_i \circ R_{-\theta_i}$, so $h_i \in F = F(l, A, P)$ and h_i lies furthermore in the kernel of the following homomorphism

$$\rho : F \rightarrow P \times P$$

$\varphi \mapsto (\varphi'(0+), \varphi'(l-))$, taking derivatives at endpoints. Let $B = \ker \rho$, so $h_i \in B$.

Theorem 3.2 (M. Stein [14]). *B' is simple and $B' = F'$*

Theorem 3.3 (K. S. Brown). *$H_*(F) \cong H_*(B) \otimes H_*(P \times P)$.*

(For a proof, see M. Stein [14].)

Let's further assume that the slope group P has rank 2, i.e. $P = \langle p, q \rangle$, $A = \mathbb{Z}[\frac{1}{p}, \frac{1}{q}]$, $p, q \in \mathbb{Z}_+$ and $d = \gcd(p-1, q-1) = 1$. $F_{p,q}, T_{p,q}$ are the corresponding groups. M. Stein, in [14], computed the homology groups of $F_{p,q}$ by using its action on a complex.

Theorem 3.4 (M. Stein). $H_1(F_{p,q})$ is a free abelian group with rank $2(d+1)$, where $d = \gcd(p-1, q-1)$.

By assumption $d = 1$, $rk_{\mathbb{Z}}(H_1(F_{p,q})) = 4$. By Theorem 3.3, $H_1(F_{p,q}) \cong H_1(B) \oplus H_1(P \times P)$ and here $P \times P \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, so $rk_{\mathbb{Z}}(P \times P) = 4 = rk_{\mathbb{Z}}(H_1(F_{p,q}))$, which implies that $H_1(B) = B/B'$ is trivial. So $B = B' = F'_{p,q}$.

So $h_i \in B = F'_{p,q}$ can be written as a product of commutators of $F_{p,q}$. So is g_i , which is conjugate to h_i in $T_{p,q}$. So overall we proved that $T_{p,q}$ is perfect, i.e. $T_{p,q} = T'_{p,q}$.

Let's compute the set $Q(T_{p,q})$. For this purpose, we need the theorem of D. Calegari [4] about the scl of elements in subgroups of $PL^+(I)$.

Theorem 3.5 (D. Calegari). Let G be a subgroup of $PL^+(I)$. Then the stable commutator length of every element of $[G, G]$ is zero.

Lemma 3.6. Let $T_{p,q} = T(1, \mathbb{Z}[\frac{1}{p}, \frac{1}{q}], \langle p, q \rangle)$ and $d = \gcd(p-1, q-1) = 1$, then $Q(T_{p,q}) = \{0\}$.

Proof: By the argument above, $f = g_1 g_2$ and $g_i = R_{-\theta_i} \circ h_i \circ R_{\theta_i}$, $i = 1, 2$. Since every homogeneous quasimorphism ϕ is a class function, it follows that $\phi(g_i) = \phi(h_i)$. Since $d = 1$, $h_i \in F'_{p,q} \subseteq PL^+(I)$ and Theorem 3.5 and Theorem 2.4 imply $\phi(h_i) = 0$, so $\phi(g_i) = 0$. For any n , suppose $f^n = g_{1n} g_{2n}$, then $\phi(g_{1n}) = \phi(g_{2n}) = 0$

$$|n\phi(f)| = |\phi(f^n)| = |\phi(g_{1n} g_{2n})| = |\phi(g_{1n} g_{2n}) - \phi(g_{1n}) - \phi(g_{2n})| \leq D(\phi)$$

so

$$|\phi(f)| \leq \frac{D(\phi)}{n}$$

for any $n \in \mathbb{Z}_+$. Let $n \rightarrow +\infty$, we get $\phi(f) = 0$. By Theorem 2.4, $Q(T_{p,q})/H^1(T_{p,q}; \mathbb{R}) = \{0\}$, but we have $T_{p,q} = T'_{p,q}$, which implies that $H^1(T_{p,q}; \mathbb{R})$ is trivial, thus $Q(T_{p,q}) = \{0\}$. \square

Theorem 3.7. Suppose $\gcd(p-1, q-1) = 1$. Let $\tilde{T}_{p,q}$ be the preimage of $T_{p,q}$ in $\text{Homeo}^+(S^1)$. Then rot is the unique homogeneous quasimorphism which sends the unit translation to 1.

Proof: By the same argument in Proposition 2.5 with Lemma 3.6 replacing the uniformly perfectness. \square

By Bavard's Theorem 2.4, if $\gcd(p-1, q-1) = 1$, then for all a in $\tilde{T}_{p,q}$ we have an equality $scl(a) = \text{rot}(a)/2D$. In the appendix we show that $D = D(\text{rot}) = 1$ on the groups $\tilde{T}_{p,q}$, and therefore this simplifies to $scl(a) = \frac{1}{2}\text{rot}(a)$. So it remains to determine the rotation numbers of elements in generalized Thompson groups.

Isabelle Liousse (in [13]) has studied the rotation numbers of elements in generalized Thompson groups, and proved the following theorem:

Theorem 3.8 (I. Liousse). Let $l \in A = \mathbb{Z}[\frac{1}{p}, \frac{1}{q}]$, $p, q \in \mathbb{Z}_+$ independent and $d = \gcd(p-1, q-1)$. Suppose d is prime or 1, then $T_{l,(p,q)} = T(l, \mathbb{Z}[\frac{1}{p}, \frac{1}{q}], \langle p, q \rangle)$ contains elements with irrational rotation numbers. In particular, if $d = 1$, $T_{l,(p,q)} \cong T_{1,(p,q)}$, for any $l \in \mathbb{Z}$.

Example 3.9 (from [13]). Let $p = 2, q = 3$, $\tilde{T}_{2,3}$ satisfies the assumption. We have the element $f \in T_{2,3}$:

$$f = \begin{cases} \frac{2}{3}x + \frac{2}{3} & \text{if } x \in [0, \frac{1}{2}] \\ \frac{4}{3}x - \frac{2}{3} & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

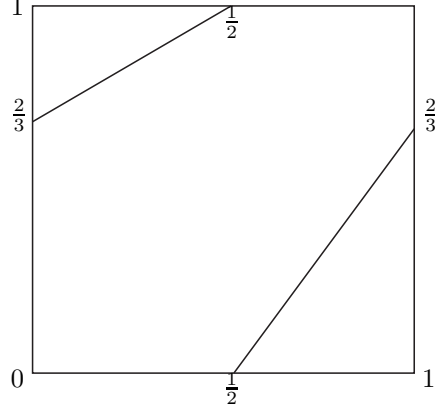


FIGURE 1.

The transformation $h(x) := 2 - \frac{1}{2^{x-1}}$, $x \in [0, 1]$ conjugates f to the rotation by ρ where $\rho = \frac{\log 3}{\log 2} - 1 \approx 0.58496250072 \dots$. Thus any lift of f has the rotation number $\frac{\log 3}{\log 2} + n$, $n \in \mathbb{Z}$. By the celebrated theorem of Gel'fand-Schneider [12], the rotation number ρ is transcendental.

f lies in the class of the homeomorphisms in $PL^+(S^1)$, satisfying the *property D* [13]. I. Lioussé showed that for any $f \in PL^+(S^1)$ with the *property D*, there exists $h \in \text{Homeo}^+(S^1)$ such that $h^{-1}fh$ is a rotation, i.e. in $SO(2)$. Thus we have a measure μ on S^1 that is invariant under f and furthermore

$$\int \log(Df) d\mu = 0$$

here Df is the function of the derivative of f , which is a piecewise constant function, so to write down the left side of the equality, we only need to get the measures of the intervals between break points. This can be obtained by the assumed *property D*, without any knowledge of the conjugate function h . This equality gives a linear equation of the rotation number of f , so in this way, we can get the rotation number.

4. APPENDIX

In this appendix, we will justify the claims in previous sections on defects of rotation numbers as homogeneous quasimorphisms. We will use a lemma by C. Bavard on defect estimation. See C. Bavard [2] or D. Calegari [5] for a proof.

Lemma 4.1 (C. Bavard). *Let ϕ be a homogeneous quasimorphism on G . Then there is an equality*

$$\sup_{a, b \in G} |\phi([a, b])| = D(\phi)$$

Proposition 4.2. $\widetilde{\text{rot}} : \widetilde{\text{Homeo}^+(S^1)} \rightarrow \mathbb{R}$ is a homogeneous quasimorphism with defect 1.

Proof: Refer to [11] for basic properties of rotation numbers.

(1) Let $f, g \in \widetilde{\text{Homeo}}^+(S^1)$. Without loss of generality, we can assume that $0 \leq f(0), g(0) < 1$. So $0 \leq f \circ g(0) < 2$. And $0 \leq \text{rot}(f) \leq 1$, $0 \leq \text{rot}(g) \leq 1$, and $0 \leq \text{rot}(f \circ g) \leq 2$. Thus we have $|\text{rot}(f \circ g) - \text{rot}(f) - \text{rot}(g)| \leq 2$, rot is a quasimorphism. That rot is homogeneous is clear from its definition.

(2) We show that $D(\text{rot}) = 1$ by using Lemma 4.1.

Take any $f, g \in \text{Homeo}^+(S^1)$ and we are going to compute $\text{rot}([f, g])$. We can still assume that $0 \leq f(0), g(0) < 1$. Suppose $0 \leq g(0) \leq f(0) < 1$, then we have by using that f, g are increasing functions:

$$g(f(0)) < g(1) = g(0) + 1 \leq f(0) + 1 \leq f(g(0)) + 1$$

so

$$f(g(0)) - g(f(0)) > -1$$

Then we have two cases:

(i) If we also have $f(g(0)) - g(f(0)) \leq 1$, then

$$-1 \leq f(g(0)) - g(f(0)) \leq 1$$

$$g(f(-1)) = g(f(0)) - 1 \leq f(g(0)) \leq g(f(0)) + 1 = g(f(1))$$

which implies

$$-1 \leq f^{-1}g^{-1}fg(0) \leq 1$$

so

$$|\text{rot}([f, g])| \leq 1$$

(ii) If instead we have $f(g(0)) > g(f(0)) + 1$, then $g(f(0)) < f(g(0)) - 1 = f(g(0) - 1) < f(0)$. Consider $H(x) = f^{-1}g^{-1}fg(x) - 1 - x$, for $x \in [0, 1]$. $H(0) = f^{-1}g^{-1}fg(0) - 1 > 0$ by assumption.

$$\begin{aligned} H(f(0)) &= f^{-1}g^{-1}fg(f(0)) - 1 - f(0) \\ &< f^{-1}g^{-1}f(f(0)) - 1 - f(0) \\ &= f^{-1}g^{-1}f^2(0) - 1 - f(0) \end{aligned}$$

We want to show that $H(f(0)) < 0$, which can be deduced from the inequality below

$$f^{-1}g^{-1}f^2(0) < 1 + f(0)$$

which is equivalent to

$$f^2(0) < g(f^2(0)) + 1$$

This is always true since $x < g(x) + 1$, for any $x \in \mathbb{R}$.

So we have $H(0) > 0$ and $H(f(0)) < 0$, here $0 < f(0) < 1$. There must be a point $y \in (0, f(0))$ such that

$$H(y) = f^{-1}g^{-1}fg(y) - 1 - y = 0$$

that is

$$f^{-1}g^{-1}fg(y) = 1 + y$$

So

$$\text{rot}[f, g] = \lim_{n \rightarrow \infty} \frac{[f, g]^n(y) - y}{n} = 1$$

The proof for the case $0 \leq f(0) \leq g(0) < 1$ is the same. Put all together and we get $D(\text{rot}) = 1$ by C. Bavard's Lemma 4.1. \square

Proposition 4.3. *Let $T = T(1, A, P)$, where $A = \mathbb{Z}[\frac{1}{n_1}, \frac{1}{n_2}, \dots, \frac{1}{n_k}]$, $P = \langle n_1, n_2, \dots, n_k \rangle$, here n_i 's are independent. Suppose $d = \gcd(n_1 - 1, \dots, n_k - 1) = 1$, then T is dense in $\text{Homeo}^+(S^1)$ with C^0 topology.*

Proof: Take an arbitrary f in $\text{Homeo}^+(S^1)$ and any $\epsilon > 0$. f is uniformly continuous, so we can choose $0 = x_0 < x_1 < \dots < x_l < 1$, x_i 's are in A that is dense in $S^1 = [0, 1]/\{0 = 1\}$, such that $|f(x_i) - f(x_{i-1})| < \frac{\epsilon}{3}$. Since A is dense in $[0, 1]$, we can find $y_0 < y_1 < \dots < y_l < 1$, y_i 's are in A and $|y_i - f(x_i)| < \frac{\epsilon}{3}$. By the Bieri-Strebel criterion (Theorem 3.1), there exists $g \in T$ such that $g(x_i) = y_i$. And from the choice of x_i 's and y_i 's, it's easy to see that $\|f - g\|_{C^0} < \epsilon$. \square

Thus we also have that $\tilde{T} \subseteq \widetilde{\text{Homeo}^+(S^1)}$ is dense. On the other hand, the map rot , thought of as a function from $\text{Homeo}^+(S^1)$ to \mathbb{R} , is continuous in the C^0 topology [11]. So we have that the defect of rotation number $D(\text{rot})$, restricted to \tilde{T} , is also 1.

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